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# Cellular automata between configuration spaces over different groups

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**Abstract.** Let $\phi : H \to G$ be a group homomorphism. A $\phi$-cellular automaton is a continuous function $\mathcal{T} : A^G \to A^H$ that is $\phi$-equivariant in the sense that $h \cdot \mathcal{T}(x) = \mathcal{T}(\phi(h) \cdot x)$ for all $h \in H$, $x \in A^G$. Each $\phi$-cellular automaton $\mathcal{T} : A^G \to A^H$ may be factorized as $\mathcal{T} = \phi^* \circ \tau$, where $\tau : A^G \to A^G$ is an id-cellular automaton and $\phi^* : A^G \to A^H$ is the pullback defined by $\phi(x) = x \circ \phi$, for all $x \in A^G$. We study under which conditions this factorization is unique, and we prove that it is when $G$ is a torsion-free abelian group. Moreover, we define the restriction of a $\phi$-cellular automaton.

**Keywords:** Cellular automata · group homomorphism · unique factorization.

## 1 Introduction

Let $G$ be a group and let $A$ be a finite set with at least two elements. The *configuration space* $A^G$ is the set of all functions of the form $x : G \to A$. This is a topological space with the *prodiscrete topology*, which is the product topology of the discrete topology of $A$. The group $G$ acts on $A^G$ via the *shift action* $\cdot : G \times A^G \to A^G$ which is defined by

$$(g \cdot x)(k) = x(g^{-1}k), \quad \forall x \in A^G,\ g, k \in G.$$

For a group homomorphism $\phi : H \to G$, a *$\phi$-cellular automaton* is defined in [1] as a function $\mathcal{T} : A^G \to A^H$ such that there exists a finite subset $T \subseteq G$, called a *memory set*, and a local map $\mu : A^T \to A$ satisfying that

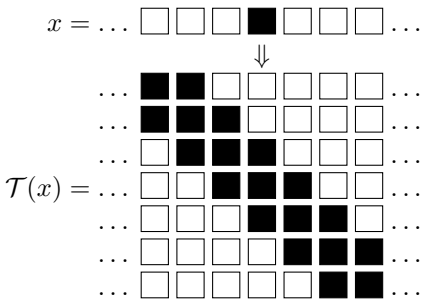
$$\mathcal{T}(x)(h) = \mu((\phi(h^{-1}) \cdot x)|_T), \quad \forall x \in A^G, h \in H.$$

The classical definition of a cellular automaton is recovered when $H = G$ and $\phi = \mathrm{id}$ is the identity homomorphism.

*Example 1.* Let $A = \{0, 1\}$. For all $x \in A^{\mathbb{Z}}$ and $(i, j) \in \mathbb{Z}^2$, define a function $\mathcal{T} : A^{\mathbb{Z}} \to A^{\mathbb{Z}^2}$ by

$$\mathcal{T}(x)(i, j) = (x(i + j - 1) + x(i + j) + x(i + j + 1)) \mod (2)$$

Then, $\mathcal{T} : A^{\mathbb{Z}} \to A^{\mathbb{Z}^2}$ is a $\phi$-cellular automaton with homomorphism $\phi : \mathbb{Z}^2 \to \mathbb{Z}$ defined by $\phi(i,j) = i + j$, for all $(i,j) \in \mathbb{Z}^2$. The following diagram shows the image of a given configuration $x \in A^{\mathbb{Z}}$, where a black box denotes a 1 and a white box denotes a 0.



The following theorem summarizes the three main results obtained in [1].

**Theorem 1.** *Let $G$ and $H$ be groups, and let $\phi : H \to G$ be a homomorphism.*

1. *A function $\mathcal{T} : A^G \to A^H$ is a $\phi$-cellular automaton if and only if $\mathcal{T}$ is continuous in the prodiscrete topologies and* $\phi$-equivariant *in the sense that*

$$h \cdot \mathcal{T}(x) = \mathcal{T}(\phi(h) \cdot x), \quad \forall h \in H, x \in A^G,$$

   *where $\cdot$ denotes the shift actions of $G$ on $A^G$ and of $H$ on $A^H$.*

2. *Let $\mathcal{T} : A^G \to A^H$ be a $\phi$-cellular automaton with memory set $T \subseteq G$ and let $\mathcal{S} : A^H \to A^K$ be a $\psi$-cellular automaton with memory set $S \subseteq H$. Then, $\mathcal{S} \circ \mathcal{T} : A^G \to A^K$ is a $(\phi \circ \psi)$-cellular automaton with memory set $\phi(S)T := \{\phi(s)t : s \in S, t \in T\}$.*

3. *A $\phi$-cellular automaton $\mathcal{T} : A^G \to A^H$ is invertible (in the sense that there exists a $\psi$-cellular automaton $\mathcal{S} : A^H \to A^G$ such that $\mathcal{T} \circ \mathcal{S} = id_{A^H}$ and $\mathcal{S} \circ \mathcal{T} = id_{A^G}$) if and only if $\mathcal{T}$ is bijective.*

This theorem generalizes three classical results on cellular automata (see [3, Ch. 1]), including the famous Curtis-Hedlund-Lyndon theorem.

**Lemma 1 ([2]).** *For any $\phi$-cellular automaton $\mathcal{T} : A^G \to A^H$ there exists a unique id-cellular automaton $\tau : A^G \to A^G$ such that*

$$\mathcal{T} = \phi^* \circ \tau,$$

*where $\phi^* : A^G \to A^H$ is defined by $\phi^*(x) = x \circ \phi$, for all $x \in A^G$.*

In the proof of Lemma 1, $\tau : A^G \to A^G$ is defined as the id-cellular automaton whose memory set and local map are the same as the ones of $\mathcal{T} : A^G \to A^H$.

A natural question is to ask when the factorization of Lemma 1 is unique. In the following section, we give some sufficient conditions for the uniqueness of this factorization.

## 2 Unique factorization

First of all, we show that for any homomorphisms $\phi, \psi : H \to G$ and id-cellular automata $\sigma, \tau : A^G \to A^G$, if $\phi^* \circ \tau = \psi^* \circ \sigma$, then $\tau = \sigma$. Let $\mu : A^T \to A$ and $\nu : A^S \to A$ be the local maps for $\tau$ and $\sigma$, respectively. By evaluating $\phi^* \circ \tau = \psi^* \circ \sigma$ on $x \in A^G$ and then on the identity element $e \in H$, we obtain

$$\mu(x|_T) = \tau(x)(e) = (\phi^*(\tau(x)))(e) = (\psi^*(\sigma(x)))(e) = \sigma(x)(e) = \nu(x|_S).$$

This implies that $\tau = \sigma$.

Therefore, the unique factorization question reduces to ask when $\phi^* \circ \tau = \psi^* \circ \tau$ implies $\phi^* = \psi^*$, which is equivalent to $\phi = \psi$.

*Example 2.*

1. If $\tau$ is a constant function, then $\phi^* \circ \tau = \psi^* \circ \tau$ for every homomorphisms $\phi$ and $\psi$.
2. If $\tau$ is surjective, then $\phi^* \circ \tau$ has a unique factorization, as $\tau$ is right-cancellative.
3. If $\phi^* \circ \tau$ is injective, then it has a unique factorization. To show this, observe that $\phi^* \circ \tau = \psi^* \circ \tau$, implies that

$$\tau(\phi(h) \cdot x) = \tau(\psi(h) \cdot x), \quad \forall h \in H, x \in A^G.$$

By injectivity, we have $\phi(h) \cdot x = \psi(h) \cdot x$ for all $h \in H, x \in A^G$. Since the shift action of $G$ on $A^G$ is faithful, we conclude that $\phi(h) = \psi(h)$ for all $h \in H$, so $\phi = \psi$.

For two homomorphisms $\phi, \psi : H \to G$, we define the *difference set* by

$$\Delta(\phi, \psi) := \{\psi(h)^{-1}\phi(h) : h \in H\} \subseteq G.$$

Note that $\Delta(\phi, \psi) = \{e\}$ if and only if $\phi = \psi$.

**Theorem 2 ([2]).** *If $\Delta(\phi, \psi)$ is infinite, then $\phi^* \circ \tau \neq \psi^* \circ \tau$ for every non-constant* id*-cellular automaton $\tau : A^G \to A^G$.*

**Corollary 1.** *If $G$ is a torsion-free abelian group, then every non-constant $\phi$-cellular automaton $\mathcal{T} : A^G \to A^H$ has a unique factorization.*

*Proof.* As $G$ is abelian, then $\Delta(\phi, \psi)$ is a subgroup of $G$. As $G$ is torsion-free, then $\Delta(\phi, \psi)$ must be infinite, so the result follows by the previous theorem.

In [4], Ville Salo has shown that the previous corollary may be generalized to an arbitrary torsion-free group $G$ using the fact that the image of an id-cellular automaton is always a *strongly irreducible subshift.*

**Theorem 3 ([2]).** *Let $G$ be an abelian or locally finite group. Then, $\Delta(\phi, \psi)$ is infinite if and only if $\phi^* \circ \tau \neq \psi^* \circ \tau$ for every non-constant* id*-cellular automaton $\tau : A^G \to A^G$.*

## 3 Restriction

For a subgroup $K \leq H$ and a homomorphism $\phi : H \to G$, let $\mathcal{T} : A^G \to A^H$ be a $\phi$-cellular automaton with memory set $T \subseteq \phi(K)$ and local function $\mu : A^T \to A$. Let $\phi|_K^{\phi(K)} : K \to \phi(K)$ be the domain and codomain restriction of $\phi$. The *restriction of $\mathcal{T}$ with respect to $K$* is the $\phi|_K^{\phi(K)}$-cellular automaton $\mathcal{T}_K : A^{\phi(K)} \to A^K$ defined by

$$\mathcal{T}_K(x)(k) := \mu((\phi(k^{-1}) \cdot x)|_T), \quad \forall x \in A^{\phi(K)},\ k \in K$$

**Lemma 2 ([2]).** *With the above notation, $\mathcal{T}_K$ is the unique $\phi|_K^{\phi(K)}$-cellular automaton such that the following diagram commutes*

$$\begin{array}{ccc} A^G & \xrightarrow{\mathcal{T}} & A^H \\ {\scriptstyle \mathrm{Res}_{\phi(K)}}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}_K} \\ A^{\phi(K)} & \xrightarrow{\mathcal{T}_K} & A^K \end{array}$$

*where $\mathrm{Res}_K : A^H \to A^K$ is defined by*

$$\mathrm{Res}_K(x) = x|_K, \quad \forall x \in A^H.$$

Induction of $\phi$-cellular automata is analogously defined in [2].

**Theorem 4 ([2]).** *Let $\mathcal{T} : A^G \to A^H$ be a $\phi$-cellular automaton with memory set contained in $\phi(K)$.*

1. *$\mathcal{T}$ is injective if and only if $\mathcal{T}_K$ is injective and $\phi$ is surjective.*
2. *$\mathcal{T}$ is bijective if and only if $\mathcal{T}_K$ is bijective and $\phi$ is bijective.*

The previous result is well-known for id-cellular automata (see [3, Ch. 1]). Moreover, it is known that an id-cellular automaton $\tau : A^G \to A^G$ is surjective if and only if $\tau_K$ is surjective. So far, it is an open question whether this result may be generalized for $\phi$-cellular automata.

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# Cellular automata generated by one pattern

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**Abstract.** Let $G$ be a group and let $A$ be a finite set with at least two elements. A cellular automaton (CA) is a transformation $\tau : A^G \to A^G$ defined via a local map $\mu : A^S \to A$, where $S$ is a finite subset of $G$, that is applied in parallel and homogeneously on $A^G$ using the shift action of $G$ on $A^G$. We say that $\tau$ is *generated by one pattern* if the identity element $e$ is in $S$ and there exists $p \in A^S$ such that $\mu(p) \neq p(e)$ and $\mu(z) = z(e)$ for all $z \in A^S \setminus \{p\}$. Such cellular automata are often idempotent; for example, we have shown that if $p$ is constant or symmetrical, then $\tau$ is idempotent. The idempotency of quasi-constant patterns has also been characterized, but it is an open problem to characterize idempotency of an arbitrary pattern. Furthermore, we have computational evidence that shows that, over a torsion-free group, non-idempotent CA always have infinite order.

**Keywords:** Cellular automata · Generating patterns · Idempotents.

## 1 Introduction

Let $G$ be a group and let $A$ be a finite set with at least two elements. The set of *configurations* $A^G$ consists of all maps from $G$ to $A$. A *pattern* (or a *block*) over a finite subset $S \subseteq G$ is a function $z : S \to A$, and the set of all patterns over $S$ is denoted by $A^S$. The *shift action* of $G$ on $A^G$ is the function $\cdot : G \times A^G \to A^G$ defined as:

$$(g \cdot x)(h) := x(g^{-1}h). \forall x \in A^G, g, h \in G.$$

A *cellular automaton* (CA) is a function $\tau : A^G \to A^G$ such that there exists a finite subset $S \subseteq G$, called a *memory set* of $\tau$, and a *local map* $\mu : A^S \to A$ such that

$$\tau(x)(g) = \mu((g^{-1} \cdot x)|_S), \quad \forall x \in A^G, g \in G.$$

The famous Curtis-Hedlund-Lyndon Theorem (see [2, Theorem 1.8.1]) establishes that a function $\tau : A^G \to A^G$ is a cellular automaton if and only if $\tau$ is $G$-*equivariant* in the sense that $\tau(g \cdot x) = g \cdot \tau(x)$, for all $g \in G$, $x \in A^G$, and $\tau$ is continuous in the *prodiscrete topology* of $A^G$ (which is the product topology of the discrete topology of $A$).

**Definition 1.** *Let $\tau : A^G \to A^G$ be a cellular automaton with memory set $S \subseteq G$ such that $e \in S$, and local defining map $\mu : A^S \to A$. We say that $\tau$ is* generated *by a pattern $p \in A^S$ if*

$$\mu(p) \neq p(e) \quad \text{and } \mu(z) = z(e),\ \forall z \in A^S \setminus \{p\}.$$

*In such case we write $\tau = \tau_p^a$ and $\mu = \mu_p^a$, where $a := \mu(p)$.*

When $A = \{0, 1\}$, we have that $a = \mu(p)$ must be equal to the complement of $p(e)$, so we may simply write $\tau_p$ and $\mu_p$ instead of $\tau_p^a$ and $\mu_p^a$, respectively.

*Example 1.* Let $G := \mathbb{Z}$, $A := \{0, 1\}$ and $S := \{-1, 0, 1\}$. Cellular automata $\tau : A^{\mathbb{Z}} \to \mathbb{Z}$ that admit a memory set $S$ are called *elementary cellular automata* (ECA) and they are labeled identified by a *Wolfram number* (see [4, Sec. 2.5]). For example, the local map of the ECA 200 is defined by the following table:

| $z \in A^S$ | 111 | 110 | 101 | 100 | 011 | 010 | 001 | 000 |
|---|---|---|---|---|---|---|---|---|
| $\mu(z) \in A$ | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |

In this table, the elements of $A^S$ are identified with triplets $z_{-1}z_0z_1 \in A^3$, where $z_i = z(i)$ for $i \in S$. The second row of the table is the binary number 11001000, which correspond to the decimal number 200. We see that this cellular automaton is generated by the pattern $p = 010$.

Denote by $X_p$ the *subshift* of $A^G$ defined by a single forbidden pattern $p : S \to A$; explicitly,

$$X_p := \{x \in A^G : (g^{-1} \cdot x)|_S \neq p,\ \forall g \in G\}.$$

For example, for $G = \mathbb{Z}$ and $A = \{0, 1\}$, the subshift $X_{11}$ is the so-called *golden mean shift* [5, Ex. 1.2.3.].

**Lemma 1.** *Let $\tau_p^a : A^G \to A^G$ be the cellular automaton generated by a pattern $p \in A^S$. Then,*

$$\mathrm{Fix}(\tau_p^a) := \{x \in A^G : \tau_p^a(x) = x\} = X_p.$$

## 2 Idempotent CA generated by one pattern

A cellular automaton $\tau : A^G \to A^G$ is *idempotent* if $\tau^2 = \tau$, which is equivalent to $\mathrm{Fix}(\tau) = \mathrm{im}(\tau)$. Hence, by Lemma 1, a cellular automaton $\tau_p^a$ is idempotent if and only if $\mathrm{im}(\tau_p^a) = X_p$.

The following results were obtained in [1].

**Theorem 1.** *Let $\tau_p^a : A^G \to A^G$ be the cellular automaton generated by a pattern $p \in A^S$. If $p$ is constant (i.e. $p(s) = p(e)$ for all $s \in S$) or $p$ is symmetrical (i.e. $S = S^{-1}$ and $p(s) = p(s^{-1})$ for all $s \in S$), then $\tau_p^a$ is idempotent.*

**Theorem 2.** *Let $\tau_p^a : A^G \to A^G$ be the cellular automaton generated by a* quasi-constant *pattern $p \in A^S$, which means that there exists $r \in S$ such that $p$ restricted to $S \setminus \{r\}$ is constant. Then $\tau_p^a$ is idempotent if and only if one of the following holds:*

1. *$a \neq p(s)$ for all $s \in S$.*
2. *$r \neq e$ and $r^2 \in S$.*
3. *$r = e$ and $S = S^{-1}$.*

The set $\mathrm{CA}(G; A)$ of all cellular automata $\tau : A^G \to A^G$ is a monoid with the composition of functions. Any group automorphism $\phi \in \mathrm{Aut}(G)$ induces an automorphism of $\mathrm{CA}(G; A)$ via conjugation by the homeomorphism $\phi^\star : A^G \to A^G$ defined by $\phi^\star(x) = x \circ \phi$, for all $x \in A^G$. Similarly, any permutation of the alphabet $\alpha \in \mathrm{Sym}(A)$ induces an automorphism of $\mathrm{CA}(G; A)$ via conjugation the invertible cellular automaton with memory set $S = \{e\}$ and local defining map $\alpha : A \to A$. We may use these automorphisms to prove the following result.

**Lemma 2.** *Let $\phi \in \mathrm{Aut}(G)$, $\alpha \in \mathrm{Sym}(A)$, and $p \in A^S$. Then, $\tau_p^a$ is idempotent if and only if $\tau_{\alpha \circ p}^{\alpha(a)}$ is idempotent if and only if $\tau_{p^\phi}^a$ is idempotent, where $p^\phi : \phi(S) \to A$ is defined by $p^\phi(\phi(s)) := p(s)$, for all $s \in S$.*

**Table 1.** Idempotency of CA defined by paterns over $S \subseteq \mathbb{Z}$.

| $S$ | Idempotent | Non-idempotent |
|---|---|---|
| $[-1, 1]$ | [000] [010] | [001] |
| $[0, 2]$ | [000] [010] | [001] [011] |
| $[-1, 2]$ | [0000] [0010] [0101] [0110] | [0001] [0011] [0100] [0111] |
| $[0, 3]$ | [0000] [0100] [0110] | [0001] [0010] [0011] [0101] [0111] |
| $[-2, 2]$ | [00000] [00010] [00100] [00110] [01001] [01010] [01110] | [00001] [00011] [00101] |
| $[0, 4]$ | [00000] [00100] [01000] [01010] [01100] [01110] | [00001] [00010] [00011] [00101] [00110] [00111] [01001] [01011] [01101] [01111] |
| $[-3, 3]$ | [0000000] [0000011] [0000100] [0000101] [0000110] [0001000] [0001011] [0001100] [0001101] [0001110] [0010001] [0010010] [0010100] [0010101] [0010110] [0011001] [0011010] [0011100] [0011101] [0011110] [0100001] [0100010] [0100110] [0101001] [0101010] [0101110] [0110001] [0110110] [0111110] | [0000001] [0000010] [0000111] [0001001] [0001010] [0010011] [0100101] |

Table 1 shows the idempotency of cellular automata generated by patterns $p \in A^S$ with $A := \{0, 1\}$ and $S \subseteq G := \mathbb{Z}$. The subset $S$ is given as an interval

$[s,t] := \{k \in \mathbb{Z} : s \leq k \leq t\}$. Let $\alpha := (1,0)$ is the permutation that transposes 0 and 1, and $\phi \in \mathrm{Aut}(\mathbb{Z})$ is the automorphism induced by multiplication by $-1$. In Table 1, the patterns are given by orbits $[p] = \{p, \alpha \circ p, p^{\phi}, (\alpha \circ p)^{\phi}\}$, when $S = S^{-1}$, and $[p] = \{p, \alpha \circ p\}$, when $S \neq S^{-1}$.

## 3 Future work

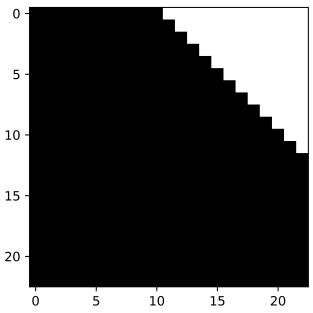


**Fig. 1.** Spacetime diagram of $\tau_{100}$.

The characterization of the idempotency of $\tau_p^a$ for an arbitrary pattern $p \in A^S$ is still an open problem.

We have shown computationally that all the non-idempotent cellular automata given by Table 1 have infinite order. This means that the cyclic subsemigroup $\langle \tau_p \rangle$ is infinite, or, equivalently, that $)(\tau_p)^m \neq (\tau_p)^n$ for all $n \neq m$. For example, if $p = 100$, we may define $x \in A^{\mathbb{Z}}$ as $x(k) = 1$ for all $k < 0$ and $x(k) = 0$ for all $k \geq 0$. The space-time diagram showing the iteration of $\tau_p$ on $x$ is given by Figure 3. Since $(\tau_p)^m(x) \neq (\tau_p)^n(x)$ for all $n \neq m$, we deduce that $\tau_p$ has infinite order.

*Conjecture 1.* Let $G$ be a torsion-free group and $p \in A^S$. If $\tau_p^a$ is not idempotent, then $\tau_p^a$ has infinite order.

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# Preliminary results of the exploration of the set of number-conserving non-uniform one-dimensional binary cellular automata with radius one and half

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**Abstract.** Until recently, it seemed that the attempt to describe all number-conserving non-uniform one-dimensional binary cellular automata with radius one and half was doomed to failure from the start due to the enormous computational complexity behind it. However, we have managed to find a tool that allows to significantly reduce the complexity of the issue, so that exploring the set of such cellular automata (at least partially) becomes possible.

**Keywords:** Non-uniform cellular automata, number conservation, finite grids.

Although in the classical formulation cellular automata (CAs) are uniform, recently scientists have increasingly turned to their non-uniform counterparts, allowing them to model phenomena that do not necessarily occur in a strictly "uniform universe". In the non-uniform CAs each single cell is allowed to have its own local updating rule (see, e.g., Pries et al. [4] or Dennunzio et al. [2]). This approach incredibly strongly increases the space of CAs under consideration. Indeed, given a one-dimensional grid of $n$ cells, we can define exactly $q^{m^q}$ uniform $q$-state $m$-input CAs and as many as $\left(q^{m^q}\right)^n$ non-uniform CAs (with the same setting) on it. For this reason, the study of non-uniform CAs is most often limited to the so-called non-uniform *elementary* CAs (ECAs), in which each cell is allowed to have its own local updating rule belonging to Wolfram's set of 256 elementary local rules (see, e.g., [3, 6]). Of course, several articles introduce more general definitions, such as the case where the cells may use different rules with different neighborhood sizes [1, 2, 5]. However, the non-uniform ECAs remain the primary focus in this research direction.

Given a set of all non-uniform $q$-state $m$-input CAs on $n$-cell grid, one would like to know which ones satisfy a given property, like for example being number-

conserving or reversible. Such a task is very challenging, as it is not possible to simply search the entire space under consideration to find those CAs that suit us. Despite these difficulties, it has recently been possible to give the detailed characterization of all number-conserving non-uniform ECAs, both on finite grids [7] and on infinite grids [8], as well as to identify those of them that are reversible.

After receiving these results, a natural question immediately arose:

*Question 1.* What do all number-conserving non-uniform one-dimensional binary CAs with radius 3/2 on finite grids look like? Which of them are reversible?

The first attempts to solve this problem did not look optimistic. To get a feeling for the level of complexity of this problem, let us note that on a grid of length $n$ as many as $(2^{16})^n = (65\,536)^n$ different non-uniform CAs can be defined. In addition, as shown by the experience gained in [7], the study should concentrate on grids with a length of at least $n = 7$. This means that if we wanted to check all of the possible rules on the smallest reasonable grid, we would need to examine $(65\,536)^7 \approx 5.2{\cdot}10^{33}$ of them. Analysing all rules is therefore clearly not a feasible approach to this problem.

However, we have recently been able to construct a directed graph $\Pi$ with the following property: there is one-to-one correspondence between the set of all length-$n$ closed directed walks in $\Pi$ and the set of all number-conserving non-uniform one-dimensional binary CAs with radius 3/2 on a finite grid with $n$ cells (assuming that $n \geq 7$). This provides us with a very powerful tool to investigate non-uniform cellular automata of this type.

Let us remind that a one-dimensional binary local rule with radius 3/2 (*i.e.*, with neighborhood size four) is any function $f : \{0,1\}^4 \to \{0,1\}$, hence, the total number of such functions is $2^{2^4} = 2^{16} = 65\,536$. The set of all such local rules will be denoted by $\mathcal{F}$. For $f \in \mathcal{F}$, its conjugation $f^{\mathrm{C}}$ and reflection $f^{\mathrm{R}}$ are defined by

$$f^{\mathrm{C}}(x,y,z,u) = 1{-}f(1{-}x,1{-}y,1{-}z,1{-}u) \quad \text{and} \quad f^{\mathrm{R}}(x,y,z,u) = f(u,z,y,x)\,.$$

Let $\mathcal{G}_n = \{0,1,\ldots,n-1\}$ denote the grid of $n$ cells, where $n$ is a positive integer. We will consider the so-called *periodic boundary conditions*: the last cell $n-1$ is adjacent to the first cell 0. Any $n$-tuple $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1}) \in X_n = \{0,1\}^n$ is called *a configuration* (of length $n$) and represents the states of the cells in $\mathcal{G}_n$: the state of cell $i \in \mathcal{G}_n$ in a configuration $\mathbf{x}$ is denoted by $x_i$.

Each rule sequence $[f_0, f_1, \ldots, f_{n-1}] \in \mathcal{F}^n$ induces a non-uniform CA $H : X_n \to X_n$ defined for any $\mathbf{x} \in X_n$ and $i \in \mathcal{G}_n$ by:

$$H(\mathbf{x})_i = f_i(x_{i-1}, x_i, x_{i+1}, x_{i+2})\,. \tag{1}$$

A non-uniform $H$ on $\mathcal{G}_n$ is called **number-conserving** if $\mu(H(\mathbf{x})) = \mu(\mathbf{x})$ for each configuration $\mathbf{x} \in X_n$, where for a configuration $\mathbf{x} \in X_n$, we denote by $\mu(\mathbf{x})$ the sum of all states in $\mathbf{x}$, *i.e.*, $\mu(\mathbf{x}) = \sum_{i \in \mathcal{G}_n} x_i$. Of course, if $H = [f_0, f_1, \ldots, f_{n-1}]$ is number-conserving, then so are $H^{\mathrm{C}} = [f_0^{\mathrm{C}}, f_1^{\mathrm{C}}, \ldots, f_{n-1}^{\mathrm{C}}]$, $H^{\mathrm{R}} = [f_{n-1}^{\mathrm{R}}, f_{n-2}^{\mathrm{R}}, \ldots, f_0^{\mathrm{R}}]$ and $H^{\mathrm{CR}} = [f_{n-1}^{\mathrm{CR}}, f_{n-2}^{\mathrm{CR}}, \ldots, f_0^{\mathrm{CR}}]$.

In [9] we have described how to generate the directed graph $\Pi = (V, E)$ with the following property: for each $n \geq 7$ there is a one-to-one correspondence between the set of all number-conserving non-uniform one-dimensional binary CAs with radius 3/2 on a finite grid with $n$ cells and the set of all length-$n$ closed directed walks in this graph (by a closed directed walk we mean a sequence of connected vertices of which the first and last are the same with repetition of edges and vertices allowed).

Each vertex $v \in V$ is some quadruple of local rules from $\mathcal{F}$ and the mentioned correspondence is as follows. If, for example, $H = [f_0, f_1, \ldots, f_6]$ is number-conserving, then in $\Pi$ there is the following closed directed walk:

$$(f_0, f_1, f_2, f_3) \to (f_1, f_2, f_3, f_4) \to (f_2, f_3, f_4, f_5) \to (f_3, f_4, f_5, f_6) \to$$

$$\to (f_4, f_5, f_6, f_0) \to (f_5, f_6, f_0, f_1) \to (f_6, f_0, f_1, f_2) \to (f_0, f_1, f_2, f_3)\,.$$

The graph $\Pi$ has exactly 273 538 vertices and 2 301 044 edges. It contains four strongly connected components $\Pi_1$, $\Pi_2$, $\Pi_3$ and $\Pi_4$ with 136 768, 136 768, 1 and 1 vertices, respectively.

Although the graph $\Pi$ is too large to be drawn in its entirety, it is small enough to be explored using the standard tools of graph theory. We hope to present some examples of such an exploration. At the present time, we are not yet able to describe all number-conserving non-uniform one-dimensional binary CAs with radius 3/2 in detail, but it looks like such a description will not be as simple as in the case of radius 1. However, the availability of the graph $\Pi$ fuels the hope that such a description will be eventually possible.

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# Advances in the parity problem for cellular automata

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**Abstract.** The parity problem is a classic benchmark for the testing of the computational power of cellular automata. In the traditional formulation, the problem involves cyclic configurations of odd length, which should converge to a fixed point of all 0s or all 1s based on the initial number of 1s, without global access to the sequence. The parity problem can be solved by a local rule with neighborhood size of 9. We are interested in finding a local rule with the smallest possible neighborhood that can correctly classify each initial configuration according to its parity.

**Keywords:** Cellular automata · parity problem · active state transition.

Cellular automata (CAs) are considered a model of distributed computing that operates without shared global memory. To explore the computational potential of CAs, various decision problems have been defined and analyzed. Among them, classification problems deserve special attention, especially in the context of binary cellular automata on finite grids with periodic boundary conditions. In these problems, the goal is to correctly classify initial configurations based on certain properties they possess.

The parity problem is one of the examples of the classification problems. This problem, in its classical formulation, is to find a CA that can properly classify each initial configuration into two classes according to its parity: if the initial configuration contains an odd number of 1s then the global state of the grid should converge to the fixed point of all 1s; otherwise, *i.e.*, if the initial configuration contains an even number of 1s then the global state of the grid should converge to the fixed point of all 0s. This problem seems to be much harder than the famous density classification problem (DCP), because the output is altered simply by a flip in any one of the input bits. Moreover, according to the formulation of the problem, it is immediately obvious that it has no sense for even-sized grids (see, for example, [1]). For this reason, the solution to this problem is considered to be a CA (actually: a local rule $f$), which correctly classifies all configurations of odd length.

It turns out that unlike DCP, the parity problem has a solution. In [1] Betel *et al.* described the local rule with radius 4 (*i.e.*, its neighborhood consists of nine cells), called BFO rule, that solves the parity problem. The question naturally arises: is there a simpler rule that can also solve the parity problem? By writing simpler, we mean that its neighborhood has less than nine cells.

In [1], a rigorous proof is given that five cells in a neighborhood is not enough to construct a local rule capable of solving the parity problem. We have investigated the case of six cells in a neighborhood, and it turns out that six cells are still not enough [2]. Unfortunately, the answer to the question of how it is for seven cells is not so simple. First of all, the method used for five and six cells is not very adaptable. While using it for five cells was rather straightforward, for six cells it became very intricate and tedious. For seven cells in a neighborhood, the complexity of the problem increases so much that this method becomes useless at all.

The authors of [1] left open the question of whether there is a rule of radius 3 that solves the parity problem. However, they have made many attempts to find such a rule (see, for example [3]). Since these attempts have always failed, some of the researchers are convinced that such a rule does not exist. However, it is worth bearing in mind that the set of all binary local rules of radius 3 is astronomically large: it has as many as $2^{128} \approx 3.4 \cdot 10^{38}$ elements. Thus, even the most advanced computer-assisted searches are unable to examine even one percent of this set.

The research we have done so far has also not given a definitive answer, but we are rather of the opinion that such a local rule of radius 3 capable of solving the parity problem exists. So far, we have found several dozens of local rules that correctly classified all configurations from a rather large test set. All these rules are defined by means of so-called active transitions and Table 1 shows one of such rules having 18 groups of active transitions. To understand this table, let us clarify the meaning of the first group. [••0**0**111] with 0 in bold means that for any values of both •, $f(\bullet\bullet0\mathbf{0}111) = 1$, that is, it is a clustered notation of the following four active transitions:

$$f(000\mathbf{0}111) = 1,\ f(010\mathbf{0}111) = 1,\ f(100\mathbf{0}111) = 1,\ f(110\mathbf{0}111) = 1.$$

The presented set of active transition groups is not the smallest. The shortest sets we found have 13 groups, but this does not at all mean that these rules are simpler for theoretical consideration.

Although we still do not have strict mathematical proof of the correctness of this local rule, we believe that it may be a solution to the parity problem.

## Acknowledgments

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| | |
|---|---|
| [••00111] | [•0110••] |
| [11100••] | [••0110•] |
| [•0000100] | [1011110] |
| [0010000] | [•1111101] |
| [0100010] | [1001000] |
| [••1010100] | [•1010100] |
| [1110110] | [•1111011] |
| [•0100010] | [0001001] |
| | [1011100] |
| | [0011101] |

**Table 1.** The LUT of a rule with radius 3, which probably solves the parity problem (the symbol • refers to any value).

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# Some Spectral Properties of D-Dimensional Cellular automata

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**Abstract**

In this paper, we show that a D-dimensional cellular automaton with equicontinuity points which is not equicontinuous admits at least a periodic factor. After that, we characterize the spectrum of D-dimensional cellular automata from the topological and the ergodic point of view where we show that cellular automata which are equicontinuous and/or with fully blocking pattern cannot have measurable irrational eigenvalues.

**Keywords:** Cellular automata, eigenvalues, ergodic theory.

## 1 Introduction

Cellular automata are employed in variety of modeling contexts. They dates back to Von Neumann in the late 1940s and the first model were biologically motivated. During the last years they have been used to model complex systems in many different domains as physics and biology.

A Cellular automaton is a discrete model of computation that evolve in space and time. It consists of a regular grid of cells that can only adopt only one given state at each time unit. For each cell, a set of cells called its neighborhood is defined relative to the specified cell and it state at each time unit is determined by applying the associated local rule of the cellular automaton to the current state of the cell and the states of the cells in which is connected. Typically, the rule for updating the state of cells is the same for each cell and does not change over time. A configuration is a snapshot of the state of all automata in the lattice. The lattice is usually $\mathbb{Z}^d$ where $d$ represents the dimension of the cellular automaton.

One way to study the dynamical properties of a cellular automaton is to endow the space of configurations with a topology and consider the cellular automaton as a discrete dynamical system and by equipping the space of configurations with the uniform Bernoulli measure [7, 8], cellular automata can be studied from the ergodic theory point of view. The uniform Bernoulli measure plays a central role in the ergodic theory of cellular automata. This measure is invariant if and only if the cellular automaton is surjective [7].

In this paper, we study the eigenvalues of D-dimensional cellular automata which formalize some kind of structure representing the nonchaotic part of the dynamics. They correspond to complex rotations that are factors. We start by showing that any $D-$dimensional cellular automaton with equicontinuity points admits at least one periodic factor. After that we characterize the spectrum of $D-$dimensional cellular automata from the topological point of view first and then we show that if the measure of the set of points of surjective cellular automaton with an eventually periodic trace (we mean by a trace of a given point $x$ studying its evolution in the central window of radius $(2r)^d$ under the iteration of the cellular automaton) is equal to one then the cellular automaton cannot have measurable irrational eigenvalues.

Furthermore, We show that if the measure of the set of equicontinuity points of $F$ is of positive measure, then either the cellular automaton $F$ have a measurable rational eigenvalue or the associated eigenfunction is null on the set of equicontinuity points. We finish by giving two applications of this result, one in the case when the cellular automaton is equicontinuous, an other one when the cellular automaton admits at least one fully blocking pattern.

# 2 Basic definitions

In this section, we present some basic notions in symbolic dynamics used along this paper.

**Notation 1**
- *$A$ is a finite alphabet.*
- *The interval $[a, b]$ represent the set of integers from $a$ to $b$.*
- *The size of a pattern $w \in A^{i_1 \times i_2 \times ... \times i_d}$ is $|w| = (i_1, i_2, ..., i_d)$.*
- *Denote by $A^{\mathbb{Z}^d}$ the set of all configurations in $\mathbb{Z}^d$ constructed over the alphabet $A$.*
- *For any pattern $w$, we define the cylinder $[w]$ at position $\overrightarrow{i} = (i_1, i_2, ..., i_d) \in \mathbb{Z}^d$ by:*

$$[w]_{\overrightarrow{i}} = \{x \in A^{\mathbb{Z}^d} : x_{\prod_{s=1}^{d} [i_s, i_s + |w|_s]} = w\}$$

- *The cylinder at the origin is denoted by $[w]_{\overrightarrow{0}}$.*

**Definition 2** *The following application define a distance over $A^{\mathbb{Z}^d}$:*

$$\forall x \neq y \in A^{\mathbb{Z}^d}, d(x, y) = 2^{-\min\{||\overrightarrow{i}||_\infty : x_{\overrightarrow{i}} \neq y_{\overrightarrow{i}}\}} \text{ where } ||\overrightarrow{i}||_\infty = \max_{s=\overline{1,D}} \{i_s\}.$$

*The space $A^{\mathbb{Z}^d}$ endowed with $d$ is compact, perfect, and totally disconnected in the product topology [6].*

- Any vector $\overrightarrow{i} \in \mathbb{Z}^d$ determines a continuous shift map $\sigma_{\overrightarrow{i}} : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ defined by $\sigma_{\overrightarrow{i}}(x)_{\overrightarrow{j}} = x_{\overrightarrow{j} + \overrightarrow{i}}, \forall \overrightarrow{j} \in \mathbb{Z}^d$.

## 2.1 Cellular Automata

**Definition 3** *Let $\Bbbk$ be a subset in $\mathbb{Z}^d$ and $f : A^{\Bbbk} \to A$ be a function called* the local rule. *A cellular automaton determined by $f$ is the function $F : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ defined by :*

$$F(x)_{\overrightarrow{m}} = f(x_{\overrightarrow{m} + \Bbbk}) \text{ for all } x \in A^{\mathbb{Z}^d} \text{ and all } \overrightarrow{m} \in \mathbb{Z}^d.$$

Curtis-Hedlund and Lyndon [5] showed that cellular automata are exactly the continuous transformations of $A^{\mathbb{Z}^d}$ that commute with all shifts. We refer to $\Bbbk$ as a radius of $F$.

**Definitions 4**
- *A point $x$ is a periodic point of period $p$ if $F^p(x) = x$ and $\forall i < p$, $F^i(x) \neq x$.*
- *If there exist an integer $m$ such that $F^m(x)$ is periodic then $x$ is called an eventually periodic point.*
- *The periodic points of the shift are called spatially periodic points and the periodic points for $F$ that are not shift periodic are called strictly temporally periodic points [3]. Any spatially periodic point is eventually periodic for the cellular automaton.*

- *A pattern $w$ of size $(k_1, k_2, ..., k_d)$ is called $(r_1, r_2, ..., r_d)-$blocking with offset $(p_1, p_2, ..., p_d)$ if there exist a non-negative integers $\{p_i\}_{i\in[1,d]}$ satisfying $p_i \leq k_i - r_i$ for all $i \in [1, d]$ such that for any $x, y \in [w]_{\overrightarrow{0}}$ and $n \geq 0$, we have: $F^n(x)_{\prod_{j=1}^{d}[p_i,p_i+r_i)} = F^n(y)_{\prod_{j=1}^{d}[p_i,p_i+r_i)}$.*

  *When the offset $p_i = 0, \forall i = i \in [1, d]$ the pattern $w$ is said fully blocking.*

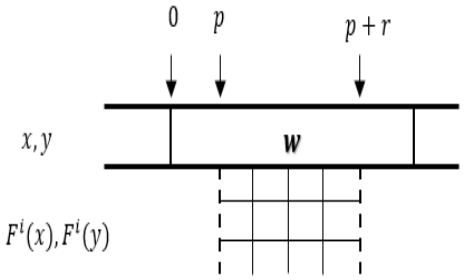


Figure 1: Blocking word, for d=1

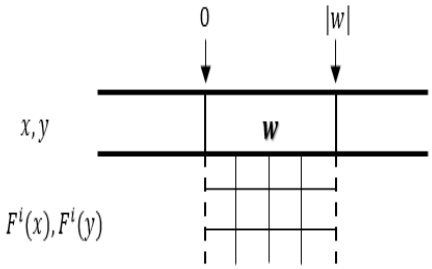


Figure 2: Fully blocking word, for d=1

- *A point $x$ is a point of equicontinuity for the cellular automaton $(A^{\mathbb{Z}^d}, F)$ if*

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall y : d(x, y) < \delta \Rightarrow \forall n \in \mathbb{N}, d(F^n(x), F^n(y)) < \varepsilon$$

- *The cellular automaton $F$ is equicontinuous if all points are points of equicontinuity and is almost equicontinuous if the set of equicontinuity contain a countable intersection of dense open sets.*

- *The cellular automaton $F$ is said to be sensitive to the initial conditions if*

$$\exists \varepsilon > 0, \forall x \in A^{\mathbb{Z}^d}, \forall \delta > 0, \exists y : d(x, y) < \delta, \exists n \in \mathbb{N} : d(F^n(x), F^n(y)) \geq \varepsilon.$$

Kurka [6] introduced a classification of one dimensional cellular automata according to sensitivity and equicontinuity of the cellular automaton.

**Theorem 5** *Let $F$ be a one dimensional cellular automaton with radius $r$. The following properties are equivalent:*

1. *$F$ is not sensitive.*
2. *$F$ admit an $r-$blocking pattern.*
3. *$F$ is almost equicontinuous.*

This equivalence do not hold in higher dimension, Gamber [2] proved that the implication $3 \Rightarrow 1$ and $1 \Rightarrow 2$ still holds in all dimensions and the existence of a fully blocking pattern implies the almost equicontinuity of the cellular automaton.

**Theorem 6 ( [2])** *Let $F : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ with radius $r$. If there exists a fully blocking pattern of size $k^d$ for $F$ where $k \geq r$ then $F$ is almost equicontinuous.*

## 2.2 Factors

Let $(A^{\mathbb{Z}^d}, F)$ and $(B^{\mathbb{Z}^d}, G)$ be two cellular automata and $\pi$ be a continuous map $\pi : A^{\mathbb{Z}^d} \to B^{\mathbb{Z}^d}$ such that $\pi \circ F = G \circ \pi$.

$$\begin{array}{ccc} A^{\mathbb{Z}^d} & \stackrel{F}{\to} & A^{\mathbb{Z}^d} \\ \pi \downarrow & & \downarrow \pi \\ B^{\mathbb{Z}^d} & \underset{G}{\to} & B^{\mathbb{Z}^d} \end{array}$$

1. If $\pi$ is bijective then we say that $\pi$ is a conjugacy and the cellular automata $(A^{\mathbb{Z}^d}, F)$,$(B^{\mathbb{Z}^d}, G)$ are conjugate.
2. If $\pi$ is surjective then we say that $\pi$ is factor map and the cellular automaton $(B^{\mathbb{Z}^d}, G)$ is a factor of $(A^{\mathbb{Z}^d}, F)$.

## 2.3 Measurable Dynamics

We endow the symbolic space $A^{\mathbb{Z}^d}$ with the sigma-algebra ß generated by all cylinder sets and $\mu$ the uniform Bernoulli measure which gives to any letter from the alphabet $A$ the same probability.
As the uniform measure is only invariant when the cellular automaton is surjective then the following $(A^{\mathbb{Z}^d}$,ß$, F, \mu)$ denote a surjective cellular automaton equipped with the uniform Bernoulli measure.

An element of the unit circle is an eigenvalue of a cellular automaton $F$ associated to the eigenfunction $g$ if we have $g \circ F = \lambda g$. Whether we place ourselves from the topological or ergodic perspective the function $g$ is required either to be continuous or measurable. An eigenvalue $\lambda = e^{2\pi i\alpha}$ is said irrational if $\alpha$ is an irrational number [8].

A cellular automaton $(A^{\mathbb{Z}^d}$,ß$, F, \mu)$ is ergodic if the measure of any invariant subset of $\mathbb{Z}^d$ is either 0 or 1. Otherwise, $F$ is ergodic if any eigenfunction is of constant module [8].

### 2.3.1 Gilman's Classification

Gilman proposed a classification for cellular automata on the basis of a measurable version of the existence of equicontinuity points. The cellular automata considred are equipped with a Bernoulli measure and are not necessarily surjective.

**Definition 7** *Let $(A^{\mathbb{Z}}, F, \mu)$ be a cellular automaton where $\mu$ is the Bernoulli measure.*
*For a point $x \in A^{\mathbb{Z}}$, we define the following set:*

$$B_{[a,b]}(x) = \{y \in A^{\mathbb{Z}} : \forall n \in \mathbb{N}, F^n_{[a,b]}(y) = F^n_{[a,b]}(x)\}$$

**Definition 8** *Let $(A^{\mathbb{Z}}, F, \mu)$ be a cellular automaton equipped with a Bernoulli measure $\mu$.*

- *A point $x \in A^{\mathbb{Z}}$ is a $\mu$- equicontinuous point if for any $m > 0$ we have:*

$$\lim_{n\to+\infty} \frac{\mu([x_{[-n,n]}] \bigcap B_{[-m,m]}(x))}{[x_{[-n,n]}]} = 1$$

- *$F$ is almost expansive if there exist an integer $m > 0$ such that for any point $x \in A^{\mathbb{Z}}$ , we have $\mu(B_{[-m,m]}(x)) = 0$.*

**Proposition 9** *Cellular automata $(A^{\mathbb{Z}}, F, \mu)$ where $\mu$ is the Bernoulli measure are divided into the three following classes:*

1. *$F \in \mathcal{A}$ if $F$ admits a topological equicontinuity point.*
2. *$F \in \mathcal{B}$ if $F$ admits a $\mu$ -equicontinuity points without any topological ones.*
3. *$F \in \mathcal{C}$ if $F$ is almost expansive.*

# 3 Results

## 3.1 Topological results

In this section, we focus on the set of periodic factors of a D-dimensional cellular automata with equicontinuity points that are not equicontinuous. After that we show that a D-dimensional cellular automaton cannot have topological irrational eigenvalues.

**Definition 10** *A fully blocking word $w$ is of order $p$ if the period of the sequence $[F^i([w]_{\overrightarrow{0}})]_{\overrightarrow{0}}$, $i \in \mathbb{N}$ is $p$. i.e.*

$$\forall x \in A^{\mathbb{Z}^d} \ , \exists m, p \in \mathbb{N} : F^{m+np}([w])_{\overrightarrow{0}} = F^m([w])_{\overrightarrow{0}}, \forall n \in \mathbb{N}.$$

**Proposition 11** *Let $(A^{\mathbb{Z}^d}, F)$ be a cellular automaton with radius $r^d$. If $F$ admit a fully blocking word $w$ of period $p \neq 1$ such that for any integer $n$ , $F^n(w)$ is fully blocking too, then $F$ has at least a non trivial periodic factor.*

**Proof.** Let $n$ be an integer and $x$ be an equicontinuity point for $F$ admitting occurrences of the blocking pattern $w$ at the coordinates given by $\overrightarrow{i} \in \mathbb{Z}^d$ such that:

$$\left\|\overrightarrow{i}\right\| = \left\|\overrightarrow{(i_1, ..., i_d)}\right\| = \max_{1 \leq j \leq d} |i_j| = nr \text{ and } \forall j \in [1, d], \ i_j \text{ is a divisor of } n.$$

The sequence of patterns $(F^i(x)_{[-nr,nr+r]^d})_{i \in \mathbb{N}}$ is eventually periodic. This is due to the density of the shift periodic configurations that have an eventually periodic trace and to the equicontinuity of $x$. Then, there exist integers $m, p$ such that for all $n \in \mathbb{N}$ :

$$F^m(x)_{[-nr,nr+r]^d} = F^{m+np}(x)_{[-nr,nr+r]^d}.$$

Consider the sets $W_k$ of cylinder sets defined by $W_k = [F^k(x)_{[-nr,nr+r]^d}]$ such that $m \leq k \leq m+p-1$ and denote by $W = \bigcup_{k=m}^{m+p-1} W_k$. By assumption, we have $\forall i \in \mathbb{N}, F^i(w)$ is a fully blocking pattern what means that $F^i(x)$ is an equicontinuity point for any integer $i$ and for $m \leq k \leq m+p-1$, we have: $F(W_k) \subseteq W_{k+1}$.

Let $(\mathbb{Z}/p\mathbb{Z}, P = (x+1) mod p)$ be a periodic dynamical system and $\pi$ be a function defined from $W$ to $\mathbb{Z}/p\mathbb{Z}$ as follow:

$$\pi(x) = k - m \text{ for } x \in W_k \text{ and } m \leq k \leq m+p-1$$

Then we have:

$$\begin{aligned} \pi(F(x)) &= \begin{cases} k+1-m \text{ if } x \in W_k \text{ st. } m \leq k \leq m+p-2 \\ 1 \text{ if } x \in W_{m+p-1} \end{cases} \\ P(\pi(x)) &= (\pi(x)+1) mod p = \begin{cases} k+1-m \text{ if } x \in W_k \text{ st. } m \leq k \leq m+p-2 \\ 1 \text{ if } x \in W_{m+p-1} \end{cases} \end{aligned}$$

By consequence, the restriction of $F$ to $W$ admit $(\mathbb{Z}/p\mathbb{Z}, P = (x+1) mod p)$ as a periodic factor of period $p$. ■

**Proposition 12** *Let $(A^{\mathbb{Z}^d}, F)$ be a cellular automaton, then $F$ cannot have a topological irrational eigenvalues.*

**Proof.** Let $\alpha$ be a real number and $g$ be a continuous non zero function such that :

$$g \circ F(x) = e^{2\pi i\alpha} g(x), \forall x \in A^{\mathbb{Z}^d}.$$

The set of periodic points for the shift is dense in $A^{\mathbb{Z}^d}$ then $g$ cannot be null on this set. So, there exist at least one shift periodic point $x$ such that $g(x) \neq 0$. Or, as the shift periodic points are eventually periodic under the iterations of $F$ then there exist integers $m, p \in \mathbb{N}$ such that: $F^m(x) = F^{m+ip}(x)$ for all $i \in \mathbb{N}$. This leads to the following:

$$g(F^{m+p}(x)) = e^{2\pi ip\alpha}.g(F^m(x))$$

What means that $e^{2\pi ip\alpha} = 1$,by consequence $\alpha$ can only be rational. ■

## 3.2 Ergodic results

In this part, we suppose that the cellular automaton is surjective and the space of the configurations is endowed with the uniform Bernoulli measure $\mu$.

**Definition 13** *Let $(A^{\mathbb{Z}^d}, F, \mu)$ be a surjective cellular automaton.*
*Define the measurable sets $Q_n$ by:*

$$\begin{aligned} Q_n &= \{x \in A^{\mathbb{Z}^d} : \exists m, p \in \mathbb{N} : \forall i \in \mathbb{N} : F^{m+ip}(x)_{[-n,n]^d} = F^m(x)_{[-n,n]^d}\} \\ &= \{x \in A^{\mathbb{Z}^d} : (F^m(x)_{[-n,n]^d})_{i\in\mathbb{N}} \text{ is eventually periodic}\}. \end{aligned}$$

$$Q_n(k) = \{x \in A^{\mathbb{Z}^d} : \exists m, p \in \mathbb{N}, 1 \leq p \leq k, \forall i \in \mathbb{N} : F^{m+ip}(x)_{[-n,n]^d} = F^m(x)_{[-n,n]^d}\}.$$

*The set $Q_n(k)$ represents the set of elements from $Q_n$ with periods bounded by $k$. $Q_n = \bigcup_{k\in\mathbb{N}} Q_n(k)$.*

**Lemma 14** *Let $(A^{\mathbb{Z}^d}, F, \mu)$ be a surjective cellular automaton.*
*If $\mu(Q_n) = 1, \forall n \in \mathbb{N}$ then $F$ cannot have measurable irrational eigenvalues.*

**Proof.** Suppose that there exist $\alpha \in \mathbb{R}$, a subset $H \subset A^{\mathbb{Z}^d}$ of measure 1 and a measurable nonzero function $g$ such that: $\forall x \in H, g(F(x)) = e^{2\pi i\alpha} g(x)$.
For $d > 0$ denote by $E_d$ the measurable set of elements $x$ of $A^{\mathbb{Z}^d}$ satisfying $|g(x)| > d$. We define by $H_n(\eta)$ for any $\eta > 0$ the measurable following set:

$$H_n(\eta) = \{x \in H : \forall y \in H, x_{_{[-n,n]^d}} = y_{_{[-n,n]^d}} \Rightarrow |g(x) - g(y)| < \eta\}.$$

By Lusin's theorem, there exist a closed set $F \subseteq H$ such that:

$$\begin{cases} \mu(H \setminus F) < \epsilon, \text{ for any } \epsilon > 0 \\ \text{Any sequence of elements } y_n \in F \text{ that converge to } x \in F,\ g(y_n) \text{ converges to } g(x) \end{cases}$$

In particular, for any point $x \in H$, consider the sequence of elements $y_n$ from $H$ that share the same coordinates with $x$ at the positions $[-n, n]^d$ , then $|g(y_n) - g(x)| < \eta$ , $\forall \eta > 0$. What means that $\mu(H_n(\eta)) = \mu(F) > 1 - \epsilon$ when $n$ tends to infinity.

We choose $d_0$ such that $\mu(E_{d_0}) = a > 0$ and take $\eta_0 < d_0$.
For $0 < \varepsilon < \frac{a}{2}$ there exist $m > 0$ such that for any $n > m$ we have:

$$\mu(H_n(\eta_0)) > 1 - \frac{\varepsilon}{3} \text{ and } \mu(Q_n) > 1 - \frac{\varepsilon}{3}$$

Set a value $n > m$ and let $k$ be an integer and let $Q_n(k)$ be the measurable part of $Q_n$ where the period is bounded by $k$. When $k$ tends to infinity $Q_n(k)$ tends to $Q_n$. In particular, for $\frac{\varepsilon}{3} > 0$ there exists then $k_0$ such that for all $k \geq k_0$: $|\mu(Q_n(k)) - \mu(Q_n)| < \frac{\epsilon}{3}$. i.e,

$$\mu(Q_n(k)) \geq \mu(Q_n) - \frac{\varepsilon}{3} \geq 1 - \frac{2\varepsilon}{3}.$$

For fixed value $k \geq k_0$ the sequence $F^k(Q_n(k))_{[-n,n]^d}$ is periodic and the period is bounded. So all the points from $F^k(Q_n(k))_{[-n,n]^d}$ have a common period $p$.

$$\forall x \in Q_n(k), \forall N \in \mathbb{N} : F^k(x)_{[-n,n]^d} = F^{k+Np}(x)_{[-n,n]^d}$$

Denote by $S = E_{d_0} \cap F^{-k}((H_n(\eta_0) \cap Q_n(k))$.
Now, we will show that $\mu(S) > 0$. We have:

$$\begin{aligned} \mu(S) &\geq \mu(E_{d_0}) + \mu(F^{-k}(H_n(\eta_0) \cap Q_n(k))) - 1 \\ &\geq \mu(E_{d_0}) + \mu(H_n(\eta_0)) + \mu(Q_n(k)) - 2 \\ &\geq a + 1 - \frac{\varepsilon}{3} + 1 - \frac{2\varepsilon}{3} - 2 \geq a - \varepsilon > \frac{a}{2}. \end{aligned}$$

Hence the set $S = E_{d_0} \cap F^{-k}(H_n(\eta_0) \cap Q_n(k))$ is of positive measure.
By definition of the set $H_n(\eta_0)$ we have:

$$\forall x \in S, \forall N \in \mathbb{N} : |g(F^k(x)) - g(F^{k+Np}(x)| < \eta_0 \tag{1}$$

Moreover,

$$\forall x \in S, \forall N \in \mathbb{N} : |g(F^k(x)) - g(F^{k+Np}(x)| = |e^{2\pi ik\alpha} - e^{2\pi i(k+Np)\alpha}||g(x)| = |1 - e^{2\pi iNp\alpha}||g(x)|$$

If $\alpha$ is irrational then the sequence $\{e^{2\pi iNp\alpha}, N \in \mathbb{N}\}$ is dense in the unite circle. There exist then $N_0$ such that $|1 - e^{2\pi iN_0p\alpha}| > 1$.
From the definition of the set $E_{d_0}$ we have $|g(x)| > d_0$ over the set $S$ and we supposed $\eta_0 < d_0$ then we obtain

$$|g(F^k(x)) - g(F^{k+N_0p}(x)| = |1 - e^{2\pi iN_0p\alpha}||g(x)| > d_0 > \eta_0 \tag{2}$$

The inequalities (1) and (2) are contradictory, then $\alpha$ cannot be irrational. ■

**Remark 15** *The the assumption of the Lemma 14 is not equivalent to the notion of $\mu-$ equicontinuity points. Indeed, the trace of any $\mu-$ equicontinuity point is eventually periodic but opposite is not valid. Furthermore, existence of $\mu-$ equicontinuity points is not sufficient to have $\mu(Q_n) = 1, \forall n \in \mathbb{N}$.*

**Proposition 16** *Let $(A^{\mathbb{Z}^d}, F, \mu)$ be a surjective cellular automaton.*
*If $F$ is equicontinuous then its eigenvalues are all rational.*

**Proof.** As the cellular automaton $F$ is surjective and equicontinuous then there exist an integer $p$ such that $F^p = I_d$ [6]. This means that $\mu(Q_n) = 1, \forall n \in \mathbb{N}$, then $F$ cannot have measurable irrational eigenvalues. ■

**Proposition 17** *Let $(A^{\mathbb{Z}^D}, F, \mu)$ be a cellular automaton. Suppose that the measure of the set of equicontinuity points $\xi$ is strictly positive and $F$ admits a measurable eigenvalue $\alpha$ associated to an eigenfunction $g$. Then, if $g(x) \neq 0, \forall x \in \xi$ , $\alpha$ cannot be irrational.*

**Proof.** Let $\alpha \in \mathbb{R}$ be an eigenvalue associated to an eigenfunction $g : A^{\mathbb{Z}^d} \to \mathbb{C}$ such that for any equicontinuity point $x \in A^{\mathbb{Z}^d}, g(x) \neq 0$ .

As the trace of any equicontinuity point $x$ is eventually periodic, this is due to the definition of equicontinuity points and the density of the shift periodic points in $A^{\mathbb{Z}^d}$.
then for any integer $n \in \mathbb{N} : \xi \subset Q_n$ . This leads to $\mu(Q_n) \geq \mu(\xi) = \beta > 0, \forall n \in \mathbb{N}$.

Consider the sets $H_n(\eta), E_d$ defined in the lemma 14 and choose $n \in \mathbb{N}$, $\eta_0 < d_0$ such that :
$\mu(E_{d_0}) = 1 - a > 0$, $\beta - a > 0$ and $\mu(H_n(\eta_0)) > 1 - \frac{\epsilon}{2}$. Denote by $S = E_{d_0} \cap F^{-k}(H_n(\eta_0) \cap Q_n(k))$ where $\mu(Q_n(k)) > \beta - \frac{\epsilon}{2}$. We have:

$$\begin{aligned} \mu(S) &\geq \mu(E_{d_0}) + \mu(F^{-k}(H_n(\eta_0) \cap Q_n(k))) - 1 \\ &\geq \mu(E_{d_0}) + \mu(H_n(\eta_0)) + \mu(Q_n(k)) - 2 \\ &\geq 1 - a + 1 - \frac{\epsilon}{2} + \beta - \frac{\epsilon}{2} - 2 > \frac{\beta - a}{2} > 0. \end{aligned}$$

From Lemma 14, $\alpha$ cannot be irrational.

■

**Corollary 18** *Let $(A^{\mathbb{Z}^D}, F, \mu)$ be a cellular automaton. If the set of equicontinuity points contains a cylinder $[u]$ then either its eigenvalue is rational or its associated eigenfunction is null on $[u]$.*

**Proposition 19**
*Let $(A^{\mathbb{Z}^D}, F, \mu)$ be a surjective cellular automaton with radius $r$.*
*If $F$ admit a fully blocking pattern $w$ then $F$ cannot have measurable irrational eigenvalues.*

**Proof.** Let $n$ be an integer. Consider the set $\tau(w, n)$ of points from $A^{\mathbb{Z}^d}$ where we will fill out with the pattern $w$ the coordinates given by $\overrightarrow{i} \in \mathbb{Z}^d$ such that

$$\left\| \overrightarrow{i} \right\| = \left\| \overrightarrow{(i_1, ..., i_d)} \right\| = \max_{1 \leq j \leq D} |i_j| = m \text{ where } n \leq m \leq n + r.$$

For any pattern $u$ such that $|u| = n \times n$, the cylinder defined over the pattern $u$ filled out with $w$ belong to $\tau(w, n)$. Thus $\mu(\tau(w, n)) > 0$ . We show that any point from $\tau(w, n)$ has an eventually periodic trace at the positions $[-n - r, n + r]^d$. Indeed, consider a point $y \in A^{\mathbb{Z}^d}$ filled outside of the coordinates $[-n - r, n + r]^d$ with the fully blocking patterns $w$ then $y$ by construction is an equicontinuity point that belong to $\tau(w, n)$ and for $\epsilon = 2^{-nr}$, there exist $\delta > 0$ such that: $\forall x \in \tau(w, n)$: $d(F^i(x), F^i(y)) \leq \epsilon$.

Recall that the values for all iterates $(F^i)_{i \in \mathbb{N}}$ of $y$ are determined just by the values of $y$ in the positions $[-n, n]^d$ then the sequence $([-n, n]^d)_{i \in \mathbb{N}}$is eventually periodic. Moreover, for a point $x$ from $\tau(w, n)$ there exist a preperiod $m$ and a period $p$ such that:

$$\forall k \in \mathbb{N} : F^m(x)|_{[-n,n]^d} = F^{m+kp}(x)|_{[-n,n]^d}$$

which means that the sequence $(F^k(x)_{[-n,n]^d})_{k \in \mathbb{N}}$ is ultimately periodic for every $x \in \tau(w, n)$.
Therefore, $\tau(w, n) \subset Q_n$ for all $n$ and $\mu(Q_n) > 0$.
From the Proposition 17, $F$ cannot have irrational measurable eigenvalues if its associated eigenfunction is not null in $\bigcap_{n \in \mathbb{N}} Q_n$. ■

# 4 Conclusion

The first result of this paper is about periodic factors of D-dimensional cellular automata where we shown that D-dimensional cellular automata with an invariant set of fully blocking patterns admit as factor at least a periodic one.

In the second part, we were interested in the spectrum of D-dimensional cellular automata where by density of the shift periodic points they cannot have topological irrational eigenvalue.

We have shown that cellular automata that are equicontinuous cannot have measurable irrational eigenvalues in all dimensions. Furthermore, If the cellular automaton admit a fully blocking patterns or the set of equicontinuity points is of positive measure then either its associated eigenvalue is rational or its eigenfunction is null on these sets. It still unknown if surjective cellular automata with equicontinuity points but without being equicontinuous can or cannot have a measurable eigenvalue which is irrational for $D \geq 2$ with respect to the uniform Bernoulli measure. What about non surjective cellular automata?. An interesting direction of future works is to know if the transitivity of a cellular automaton implies the mixing property or not.

We know that transitive dynamical systems on the interval are mixing, is it possible to think that dimension matter here, in other words are one dimensional transitive cellular automata mixing ? Is it possible that this is not true in higher dimension ?.

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# 3D Cellular Model of CO2 Spread With Random Walk

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**Abstract.** We investigate a new method for the spread of the CO2 particles. First, we model the movement of the CO2 particles by using random walk in 3D and a Cell-DEVS. We show different simulations for different indoor floor plans using the Cadmium Cell-DEVS library. The objective is to show how to integrate Cell-DEVS and random walk models for the spread of airborne particles in indoors environments. We discuss how to achieve these goals and the development of tools to build such models.

**Keywords:** CO2 spread, random walk, Cell-DEVS.

## 1 Introduction

Discrete event systems specifications (DEVS) are popular to build models of complex systems, and the Cell-DEVS extension allowed to build cellular models in discrete-event, using a formal specification and an abstract simulation algorithm. DEVS uses basic "atomic" models that may be hierarchically connected together and form a DEVS coupled composite. In DEVS, discrete events make the model evolve upon certain events external events or the execution of an internal time delay function. The formalism of a DEVS atomic model is as follows [1].

$$M = < X, Y, S, \delta_{int}, \delta_{ext}, \lambda, ta > \quad (1)$$

X, Y, and S are set of input, output, and states, respectively. The $\delta_{int}$ and $\delta_{ext}$ are internal and external transition functions in turn. The *ta* is the time advance function defining the duration of the state and $\lambda$ is the output function that is triggered when the *ta* expires. After the expiration, the state of the model changes according to the $\delta_{int}$ after executing the output function.

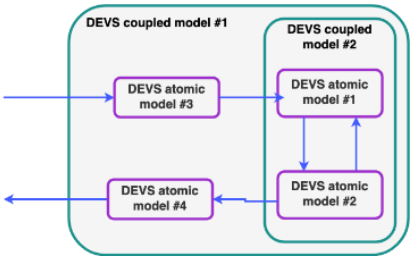


**Fig. 1.** A simple DEVS coupled model

Figure 1 shows a composite of DEVS models, called a coupled model; which includes atomic and coupled models constructed hierarchically. These submodels can communicate with each other through I/O ports. DEVS was extended to build cellular models specification such as neighboring cell parameters, border cells parameter, and the size of the cell. The CELL-DEVS coupled model is as follows [1]:

$$ECA = < S, n, \{t_1, \ldots, t_n\}, C, N, B, T, \tau, q. \boldsymbol{Z_0}^{+} > \quad (2)$$

In statement (2), the term *S* is a set of states, *n* are the dimensions of the space, *{t}* is the number of cells on each of the *n* dimensions, *T* is a global transition function, a composite of $\tau$ local computing functions that are activated upon receiving an input. The *B* is a parameter to define the border shape, and *N* are inputs coming from neighboring cells. There are several software and libraries to simulate the Cell-DEVS models; we used the Cadmium tool, in which we define a *local_computation* ($\tau$) and an *output_delay* for each cell.

We have used these methods to propose a new model of spread of CO2 particles indoor, which is built as a combination of random walk and Cell-DEVS. The following section discusses how to build such a model. The model was defined in order to study the diffusion of CO2 particles that was used during the COVID-19 pandemic to predict the chances of catching the virus indoors [4, 5], as well of the effectiveness of face masks in preventing airborne transmission of viruses [3]. The model presented here shows a different method to achieve such goals.

## 2 Building a 3D Random-Walk model with Cell-DEVS

A closed space was structured as a set of neighboring cells representing the 3D layout of a room. Each cell was comprised of one of six options, each representing CO2 levels measured in particles per million (ppm) based on a set of gas diffusion rules: (0) open-air space (constant 500 ppm CO2 level); (1) CO2 source (fixed amount of CO2 added at the average breathing interval (AVI)); (2) walls (no CO2 diffusion); (3) open door, (4) window, and (5) vent (CO2 diffusers with backgrounds of 500, 400, and <300 ppm respectively). Each cell's behavior was determined by its underlying model, which had a set of basic rules about its behavior and its effect on neighboring cells. Notably, diffuser cells could not be affected by their neighbors' increased levels of CO2.

The next step in the model was to define the transition function of each cell to the random walk algorithm. In a random walk, each object can move freely in all directions. In the 2D version of a random walk, as shown in figure 2a, the objects can move in four directions up, down, left, and right. Similarly in the 3D version, the object can also move upward and downward. The probability of choosing the direction is varied and can be changed in each step. The specific parameters of each cell need to be defined. This can be done by adding these parameters to a configuration file in JSON (Figure 2c), including the types of objects such as doors, walls, windows, vents, CO2 sources, as well as specific parameters such as concentration, or concentration increases.

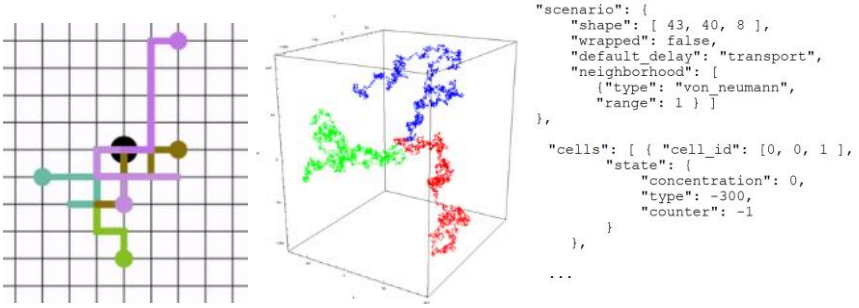


**Fig. 2.** Random Walk models

CO2 sources increase the concentration of CO2, and the vents, doors, and windows decrease the concentration. To do so, CO2 particles move to the neighbor cells and increase their CO2 concentration. To add another dimension to the model, two more neighbors need to be defined in the model. To create different scenarios for simulation we designed a 3D web application as shown in figure 3.

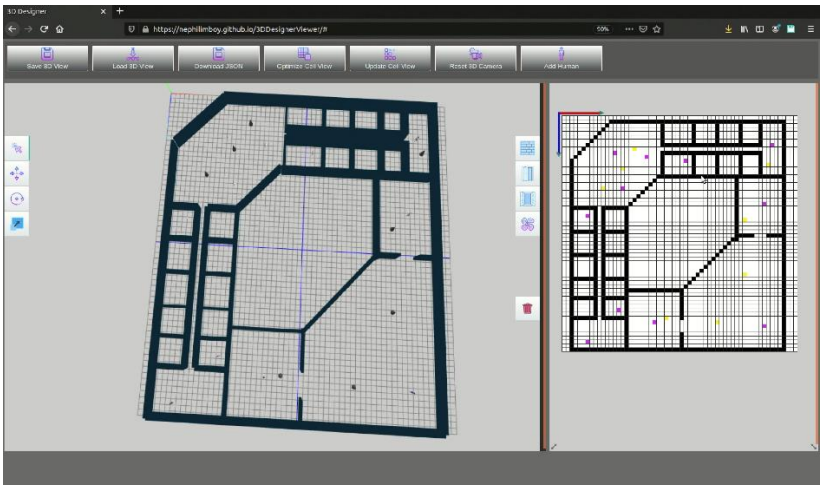

**Fig. 3.** A web application to design scenarios

$CO_2$ producers were set in different areas of empty rooms. After 30 minutes of simulation, the entire room was saturated with high levels of $CO_2$. Next, the layout of the room was modified to include vents. Once the distribution of $CO_2$ within the model was verified, diffusion cells in different configurations were added to investigate varying distribution and diffusion patterns, allowing for the analysis of $CO_2$ sensor placement. Two $CO_2$ sensor cells were added to both the left and right walls of the cubical room configuration boasting a window, door, and vent. A single $CO_2$ generator was placed in the right cubical. The $CO_2$ levels of the sensor cells were analyzed against time to see the level of $CO_2$ that reached them along with the time it took to detect a significant increase. As described above in the one generator in a cubicle simulation,

the walls, now bolstered by ventilation units, kept the $CO_2$ from spreading to the other side of the room. The simulation execution with random walk is faster than using a deterministic model, as each particle can move freely as seen in the following 3D simulation (seen in 2D in Figure 4).

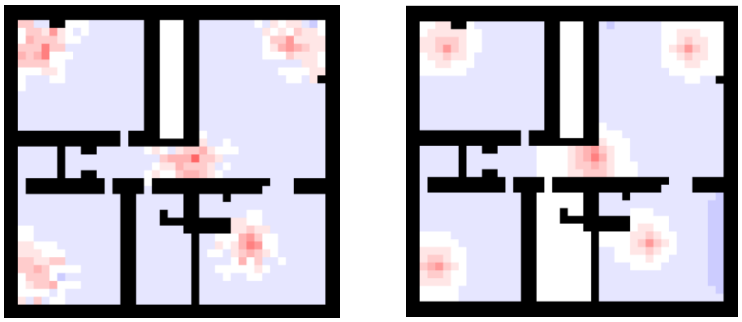

**Fig. 4.** Comparing random walk models and deterministic models

## 3. Conclusion

We introduced a CO2 model in 3D space with random walk using the Cell-DEVS formalism. This cellular model allowed defining the movement of CO2 particles and a simulation more realistic when compared with existing models. The addition of a dynamic element to a model of $CO_2$ dissipation in a room yielded interesting results and offers a range of new possibilities for future simulations. The first is the new possibility of adding variable or simply different breathing rates to the simulation. Another is the scheduled movement of one or many $CO_2$ generators within an environment that can be used to simulate the dispersion of $CO_2$ within a workplace or home with different options for the movement of individuals.

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# The BFO rule really works!

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**Abstract.** In 2013, the paper by Betel, Oliveira and Flocchini was published, which presented a binary local rule, called BFO, after the surnames of its authors, that solves the parity problem. However, in 2023, 10 years later, a counterexample emerged that the BFO rule does not work; moreover, speculation began that perhaps cellular automata are not able to solve the parity problem in the classical way at all. During our presentation, we hope to convince everyone that it is as in the title.

**Keywords:** Cellular automata, parity problem, BFO rule

One of the most frequently studied decision problems is the parity problem. In its classical formulation, this problem consists of finding a CA that can properly classify each initial configuration into two classes according to its parity: if the initial configuration contains an odd number of 1s, then the global state of the grid should converge to the fixed point of all 1s; otherwise, *i.e.*, if the initial configuration contains an even number of 1s, then the global state of the grid should converge to the fixed point of all 0s. However, according to the formulation of the problem, it is immediately clear that it makes no sense for even-sized grids. For this reason, the solution to this problem is considered to be a CA (actually, a local rule $f$) that correctly classifies all configurations of odd length.

In 2013 it appeared, that there exists such a local rule that can solve the parity problem. In fact, such a rule has been described in Betel *et al.* [1] and called BFO. This local rule has radius 4 (*i.e.*, its neighborhood consists of nine cells) and, in general, the idea underlying the BFO rule design is to reduce the number of blocks of 0s and 1s present in the initial configuration. This is accomplished by propagating blocks of 1s to the right side, two cells per iteration, combined with the simultaneous propagation of 0s to the left. Actually, the BFO rule is much more complicated, as it must include the appropriate stopping conditions for these two major trends to coexist and it can be described as follows (the original formulation from [1]):

– *Rightward growth of 1-blocks* a singleton 1 grows to the right, by two 1s at each step, if it is preceded and followed by at least two 0s, as prescribed by transitions $T_3$ and $T_4$. A block of three or more 1s grows to the right if it is

followed (on the right) by at least two 0s. This behavior is created by the pair of transitions $T_1$ and $T_2$.
- *Annihilation of pairs of 1s* as a consequence of transitions $T_5$ and $T_6$, an isolated pair of 1s is always eliminated.
- *Leftward growth of 0-blocks* a (01) block moves to the left, leading a growing block of 0s (at a growth rate of two 0s per step) if there are at least three 1s to its left and one of the following: (i) at least three 1s to the right of the 0 (the growth is obtained by the pair of transitions, $(T_9, T_{11})$; or (ii) at least one 0 to its right (due to $T_9$ and $T_{10}$). Note that the pair $(T_9, T_{11})$ starts the growth of a 0-block, while the pair $(T_9, T_{10})$ continues the growth as far as possible.
- *Local shift* a (101) block is transformed into (110) if there are a 0 on its left and at least two 0s on its right (combination of transitions $T_7$ and $T_8$).
- *Local adjustment* finally, if a (0110) block is preceded by at least three 1s that is, $(\ldots 1110110\ldots)$ occurs, in order to avoid parity errors due to the annihilation of the pair of 1s, we force the creation of a solid block of 0s to the right of the existing block of 1s with transition pair $(T_9, T_{12})$, so that $(\ldots 1110110\ldots)$ becomes $(\ldots 1000000\ldots)$.

Since the LUT of the BFO rule contains as many as 512 values, it is not worth specifying it in full, but only the so-called *active transitions* (ATs) are indicated, as in Table 1.

| | |
|---|---|
| $T_1 : [\bullet 111\mathbf{0}0\bullet\bullet\bullet]$ | $T_5 : [\bullet\bullet\bullet 0\mathbf{1}10\bullet\bullet]$ |
| $T_2 : [1110\mathbf{0}\bullet\bullet\bullet\bullet]$ | $T_6 : [\bullet\bullet 011\mathbf{0}\bullet\bullet\bullet]$ |
| $T_3 : [\bullet 001\mathbf{0}0\bullet\bullet\bullet]$ | $T_8 : [\bullet 010\mathbf{1}00\bullet\bullet]$ |
| $T_4 : [0010\mathbf{0}\bullet\bullet\bullet\bullet]$ | $T_9 : [\bullet\bullet\bullet 1\mathbf{1}101\bullet]$ |
| $T_7 : [\bullet\bullet 01\mathbf{0}100\bullet]$ | $T_{10} : [1110\mathbf{1}0\bullet\bullet\bullet]$ |
| | $T_{11} : [1110\mathbf{1}11\bullet\bullet]$ |
| | $T_{12} : [\bullet\bullet 11\mathbf{1}0110]$ |

Table 1: The LUT of the BFO rule (the symbol $\bullet$ refers to any value).

Unfortunately, in the description of BFO there is a slight inaccuracy. For this reason, if someone simply wants to use Table 1 to implement the BFO rule on a computer, that person will very quickly find a counterexample of length 13,

namely 0001110101001, which is a periodic configuration for BFO. But, fortunately, the authors of [1] have described in great detail the idea of BFO action and based on this description, the mentioned inaccuracy is easily corrected. Namely, the sentence "*Local shift* a (101) block is transformed into (110) if there are a 0 on its left and at least two 0s on its right (combination of transitions $T_7$ and $T_8$)." should be "*Local shift* a (101) block is transformed into (011) if there are at least two 0s on its left and a 0 on its right (combination of transitions $T_7$ and $T_8$)" and consequently the transitions $T_7$ and $T_8$ should be properly redefined, as in Table 2.

| | |
|---|---|
| $T_1 : [\bullet 111\mathbf{0}0\bullet\bullet\bullet]$ | $T_5 : [\bullet\bullet\bullet 0\mathbf{1}110\bullet\bullet]$ |
| $T_2 : [1110\mathbf{0}\bullet\bullet\bullet\bullet]$ | $T_6 : [\bullet\bullet 01\mathbf{1}10\bullet\bullet\bullet]$ |
| $T_3 : [\bullet 001\mathbf{0}0\bullet\bullet\bullet]$ | $T_8 : [\bullet\bullet 001\mathbf{0}10\bullet]$ |
| $T_4 : [0010\mathbf{0}\bullet\bullet\bullet\bullet]$ | $T_9 : [\bullet\bullet\bullet 1\mathbf{1}101\bullet]$ |
| $T_7 : [\bullet 001\mathbf{0}10\bullet\bullet]$ | $T_{10} : [1110\mathbf{1}0\bullet\bullet\bullet]$ |
| | $T_{11} : [1110\mathbf{1}11\bullet\bullet]$ |
| | $T_{12} : [\bullet\bullet 11\mathbf{1}0110]$ |

Table 2: The LUT of the BFO rule with the correction (the symbol $\bullet$ refers to any value).

In order to convince everyone that the BFO rule defined in Table 2 really solves the parity problem, we will try to provide a rigorous mathematical proof.

## References

[1] Heather Betel, Pedro P. Oliveira, and Paola Flocchini. "Solving the Parity Problem in One-Dimensional Cellular Automata". In: *Natural Computing: An International Journal* 12.3 (Sept. 2013), pp. 323–337. ISSN: 1567-7818. DOI: 10.1007/s11047-013-9374-9.